

MORE ON ALMOST SOUSLIN KUREPA TREES

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ABSTRACT. It is consistent that there exists a Souslin tree T such that after forcing with it, T becomes an almost Souslin Kurepa tree. This answers a question of Zakrzewski [6].

1. INTRODUCTION

In this paper we continue our study of ω_1 -trees started in [3] and prove another consistency result concerning them. Let T be a normal ω_1 -tree. Let's recall that:

- T is a Kurepa tree if it has at least ω_2 -many branches.
- T is a Souslin tree if it has no uncountable antichains (and hence no branches).
- T is an almost Souslin tree if for any antichain $X \subseteq T$, the set $S_X = \{ht(x) : x \in X\}$ is not stationary (see [1], [6]).

We refer to [3] and [4] for historical information and more details on trees.

In [6], Zakrzewski asked some questions concerning the existence of almost Souslin Kurepa trees. In [3] we answered two of these questions but one of them remained open:

Question 1.1. *Does there exist a Souslin tree T such that for each G which is T -generic over V , T is an almost Souslin Kurepa tree in $V[G]$?*

In this paper we give an affirmative answer to this question.

Theorem 1.2. *It is consistent that there exists a Souslin tree T such that for each G which is T -generic over V , T is an almost Souslin Kurepa tree in $V[G]$.*

The rest of this paper is devoted to the proof of this theorem. Our proof is motivated by [2] and [3].

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2. PROOF OF THEOREM 1.2

Let V be a model of $ZFC + GCH$. Working in V we define a forcing notion which adds a Souslin tree which is almost Kurepa, in the sense that T becomes a Kurepa tree in its generic extension. The forcing notion is essentially the forcing notion introduced in [2] and we will recall it here for our later usage. Conditions p in \mathbb{S} are of the form $\langle t, \langle \pi_\alpha : \alpha \in I \rangle \rangle$, where we write $t = t_p$, $I = I_p$ and $\langle \pi_\alpha : \alpha \in I \rangle = \bar{\pi}^p$ such that:

- (1) t is a normal ω -splitting tree of countable height η , where η is either a limit of limit ordinals or the successor of a limit ordinal. We denote η by η_p .
- (2) I is a countable subset of ω_2 .
- (3) Every π_α is an automorphism of $t \restriction Lim$, where Lim is the set of countable limit ordinals and $t \restriction Lim$ is obtained from t by restricting its levels to Lim .

The ordering is the natural one: $\langle s, \bar{\sigma} \rangle \leq \langle t, \bar{\pi} \rangle$ iff s end extends t , $dom(\bar{\sigma}) \supseteq dom(\bar{\pi})$ and for all $\alpha \in dom(\bar{\pi})$, $\sigma_\alpha \restriction t = \pi_\alpha$.

Remark 2.1. In [2], the conditions in \mathbb{S} must satisfy an additional requirement that we do not impose here. This is needed in [2] to ensure the generic T is rigid. Its exclusion does not affect our proof, and in fact simplifies several details.

Let

$$\mathbb{P} = \{p \in \mathbb{S} : \text{for some } \alpha_p, \eta_p = \alpha_p + 1\}.$$

It is easily seen that \mathbb{P} is dense in \mathbb{S} . Let G be \mathbb{P} -generic over V . Let

$$T = \bigcup \{t_p : p \in G\}$$

and for each $\alpha < \omega_2$ set

$$\pi_i = \bigcup \{\sigma_i : \exists u = \langle t, \bar{\sigma} \rangle \in G, i \in I_u\}.$$

Then (see [2], Lemmas 2.3, 2.7, 2.9 and 2.14):

Lemma 2.2. (a) \mathbb{P} is ω_1 -closed and satisfies the ω_2 -c.c.,

(b) $T = \langle \omega_1, <_T \rangle$ is a Souslin tree.

(c) Each π_i is an automorphism of $T \restriction Lim$.

(d) If b is a branch of T , which is T -generic over $V[G]$, and if $b_i = \pi_i \text{``} b$, $i < \omega_2$, then the b_i 's are distinct branches of T . In particular T is almost Kurepa.

Let $S = \{\alpha_p : p \in G, \alpha_p = \bigcup \{\alpha_q : q \in G, \alpha_q < \alpha_p\}\}$ and $I_p = \bigcup \{I_q : q \in G, \alpha_q < \alpha_p\}$.

Then as in [3], Lemma 2.4, we can prove the following:

Lemma 2.3. *S is a stationary subset of ω_1 .*

Working in $V[G]$ let \mathbb{Q} be the usual forcing notion for adding a club subset of S using countable conditions and let H be \mathbb{Q} -generic over $V[G]$. Then (see [5] Theorem 23.8):

Lemma 2.4. (a) \mathbb{Q} is ω_1 -distributive and satisfies the ω_2 -c.c.,

(b) $C = \bigcup H \subseteq S$ is a club subset of ω_1 .

Let

$$\mathbb{R} = \{\langle p, \check{c} \rangle : p \in \mathbb{P}, p \Vdash \check{c} \in \mathbb{Q} \text{ and } \max(c) \leq \alpha_p\}.$$

Since \mathbb{P} is ω_1 -closed, $\mathbb{Q} \subseteq V$ and hence we can easily show that \mathbb{R} is dense in $\mathbb{P} * \mathbb{Q}$.

Lemma 2.5. *T remains a Souslin tree in $V[G][H]$.*

Proof. We work with \mathbb{R} instead of $\mathbb{P} * \mathbb{Q}$. Let \mathcal{A} be an \mathbb{R} -name, $r_0 \in \mathbb{R}$ and $r_0 \Vdash \mathcal{A}$ is a maximal antichain in \mathcal{T} . Let \check{f} be a name for a function that maps each countable ordinal α to the smallest ordinal in $\mathcal{A}[G * H]$ compatible with α . Then as in [2] we can define a decreasing sequence $\langle r_n : n < \omega \rangle$ of conditions in \mathbb{R} such that

- r_0 is as defined above,
- $r_n = \langle p_n, \check{c}_n \rangle = \langle \langle t_n, \vec{\pi}^n \rangle, \check{c}_n \rangle$,
- $\alpha_{p_n} < \alpha_{p_{n+1}}$,
- r_{n+1} decides $\check{f} \restriction t_n$, say it forces “ $\check{f} \restriction t_n = \check{f}_n$ ”,
- $r_{n+1} \Vdash \check{C} \cap (\alpha_{p_n}, \alpha_{p_{n+1}}) \neq \emptyset$,

Let $p = \langle t, \vec{\pi} \rangle$ where $t = \bigcup_{n < \omega} t_n$, $\text{dom}(\vec{\pi}) = \bigcup_{n < \omega} \text{dom}(\vec{\pi}^n)$ and for $i \in \text{dom}(\vec{\pi})$, $\pi_i = \bigcup_{n < \omega} \pi_i^n$. Let $c = \bigcup_{n < \omega} c_n \cup \{\alpha_p\}$, where $\alpha_p = \sup_{n < \omega} \alpha_{p_n}$. Then $p \in \mathbb{S}$, but it is not clear that $p \Vdash \check{c} \in \mathbb{Q}$.

Let $f = \bigcup_{n < \omega} f_n$ and set $a = \text{ran}(f \restriction t)$. As in [2], Lemma 2.9, we can define a condition $s = \langle q, \check{c} \rangle$ such that

- $s \in \mathbb{R}$,
- $\eta_q = \alpha_p + 1$, (and hence $\alpha_q = \alpha_p$),

- $s \parallel - \text{“}\mathcal{A} \cap \check{t} \text{ is a maximal antichain in } \check{t}\text{”}$,
- Every new node (i.e. every node at the α_p -th level) of the tree part of s is above a condition in a .

It is now clear that $s \parallel - \mathcal{A} = \check{a}$, and hence $s \parallel - \mathcal{A}$ is countable”. The lemma follows. \square

From now on we work in $V^* = V[G][H]$. Thus in V^* we have a Souslin tree T . We claim that T is as required. To see this force with T over V^* and let b be a branch of T which is T -generic over V^* .

Lemma 2.6. *In $V^*[b]$, T is an almost Souslin Kurepa tree.*

Proof. Work in $V^*[b]$. By Lemma 2.2(d) T is a Kurepa tree. We now show that T is almost Souslin. We may suppose that T is obtained using the branches b and $b_i, i < \omega_2$, in the sense that for each $\alpha < \omega_1$, T_α , the α -th level of T , is equal to $\{b(\alpha)\} \cup \{b_i(\alpha) : i < \omega_2\}$ where $b(\alpha)$ ($b_i(\alpha)$) is the unique node in $b \cap T_\alpha$ ($b_i \cap T_\alpha$). We further suppose that $b = b_0$.

Now let $\alpha \in C$, and let $p \in G$ be such that $\alpha = \alpha_p$. We define a function g_α on T_α as follows. Note that $T_\alpha = \{b_i(\alpha) : i \in I_p\}$. Let

$$g_\alpha(b_i(\alpha)) = b_i(\alpha_q)$$

where $q \in G$ is such that $\alpha_q < \alpha$ is the least such that $i \in I_q$ (such a q exists using the fact that $C \subseteq S$). It is easily seen that g_α is well-defined (it does not depend on the choice of p), and that for each $x \in T_\alpha$, $g_\alpha(x) <_T x$. The rest of the proof of the fact that T is almost Souslin is essentially the same as in [3], Lemma 2.6. \square

This concludes the proof of Theorem 1.2.

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